Quantum Phase Estimation

Convert an eigenvalue phase into a binary estimate using controlled powers and an inverse Fourier transform.

Undergraduate 22 min lesson Dr. Charith Jayasekara

By the end, you'll be able to

- State the eigenstate and controlled-power input assumptions.
- Derive the exact three-eighths phase example.
- Separate finite resolution from physical noise and shot uncertainty.

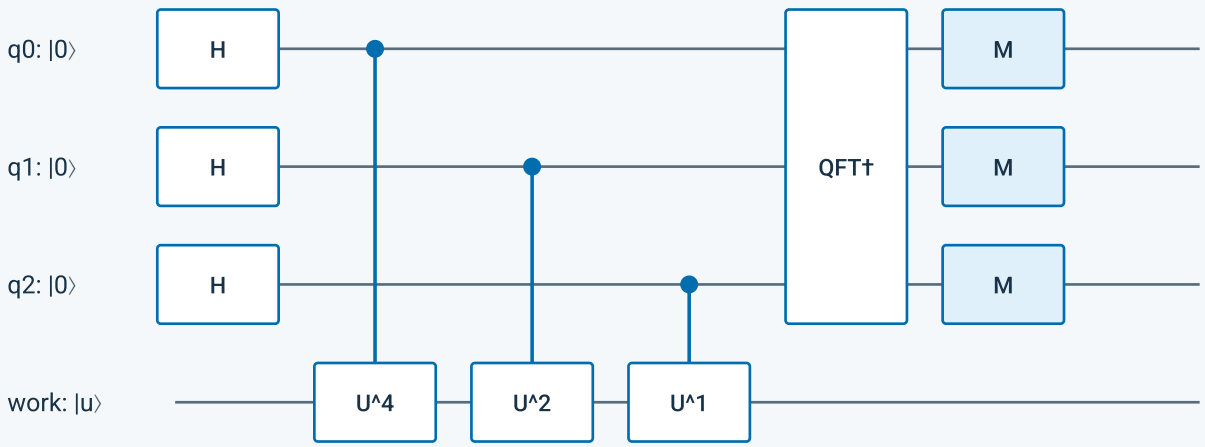
Read the phase of an eigenvalue

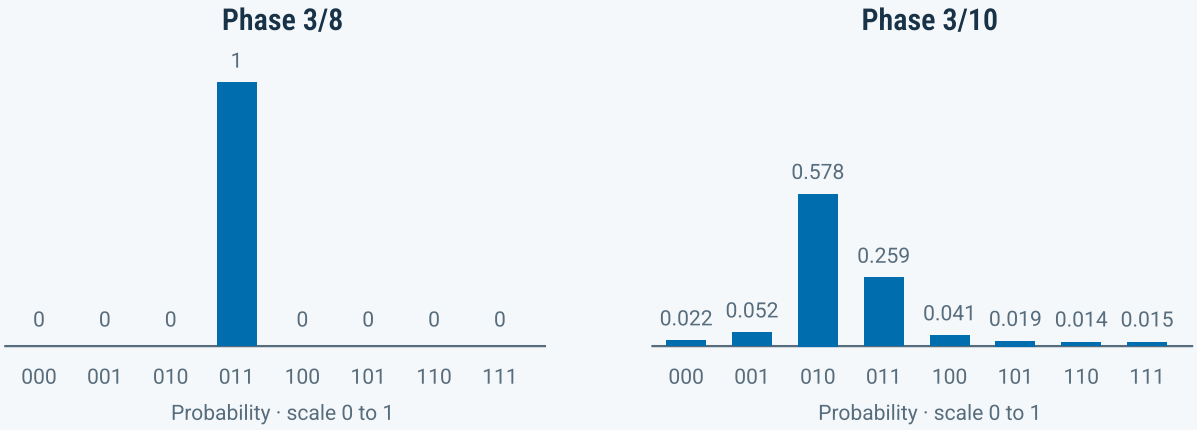
Quantum phase estimation, or QPE, estimates the phase associated with an eigenstate of a unitary operation. If applying U to a state returns that same state multiplied by a unit complex number, the state is an eigenstate and that number is its eigenvalue. QPE turns the otherwise global-looking eigenvalue phase into a relative pattern on a counting register, then reads it using an inverse Fourier transform.

Review phase kickback and the QFT. The important distinction is between a phase on an isolated whole state and a phase applied conditionally to one branch of a control. Conditional use makes the relationship observable.

WORKED DIAGRAM · A12

Phase estimation: exact and off-grid phases





Controlled U⁴, U² and U write the eigenphase onto three counting bits. Inverse QFT returns 011 exactly for φ=3/8. For φ=0.3, the ideal probability already spreads across grid labels; this spread is independent of hardware noise.

What goes in, what comes out, what we assume

You receive a unitary U and an eigenstate u satisfying

$$U|u\rangle = e^{2\pi i\phi}|u\rangle, \qquad 0 \le \phi < 1.$$

The desired output is a classical approximation to phi. The basic circuit assumes controlled access to powers of U and enough coherence to implement them. It also assumes a suitable input eigenstate, or accepts a superposition and interprets the resulting eigenphase mixture. Preparing a useful eigenstate can be a major part of a real application.

The number phi is a fraction of a full turn. It is not an angle in radians until multiplied by two pi. A value of three eighths means three pi over four radians. Keep that conversion separate from the browser controls, which may express some other phase parameters in degrees.

Build the circuit

Use t counting qubits initially zero and a work register in u. Apply H to every counting qubit. In numeric leftmost-most-significant order, the counting bits control powers of U so that basis label k applies U to the power k. The work register remains u, while the counting component receives phase exp of two pi i k phi.

Apply the inverse QFT to the counting register and measure it. If the result is j, the basic estimate is j divided by two to the power t. When phi is exactly on that grid, the ideal result is exact. Otherwise, probabilities concentrate near the phase but spread over several labels.

Controlled powers write the eigenphase as a regular phase pattern across the counting labels.

$$\frac{1}{\sqrt{L}}\sum_{k=0}^{L-1}|k\rangle|u\rangle \longmapsto \frac{1}{\sqrt{L}}\sum_{k=0}^{L-1}e^{2\pi i k\phi}|k\rangle|u\rangle, \qquad L=2^t.$$

t is the number of counting qubits. With the leftmost counting bit most significant, its controlled power is $U^{2^{t-1}}$, down to U on the rightmost bit.

Worked phase three eighths

Take t equal to three and phi equal to three eighths. After the controlled powers, the counting state has eight equal-magnitude components with phase step three pi over four. This is exactly the positive-exponent QFT of basis label three. Applying the inverse QFT therefore gives 011 with certainty under numeric three-bit ordering.

The work register remains the eigenstate. It has not been measured to reveal its full amplitudes. The algorithm has learned one eigenphase property of the operation on that state. This is useful in larger tasks because the phase may encode an energy, a period or a success fraction.

You can also inspect each counting qubit’s phase contribution before the inverse transform. The most significant bit controls U to the fourth power, contributing a phase determined by four phi modulo one. The next controls U squared and the last U. The inverse Fourier transform combines their correlated phase information into a binary label.

The exact-grid phase pattern is a Fourier basis column, so the inverse transform recognises one label.

$$\phi=\frac{3}{8},\quad t=3: \qquad \frac{1}{\sqrt{8}}\sum_{k=0}^{7}e^{2\pi i\,3k/8}|k\rangle = F_8|3\rangle \xrightarrow{F_8^\dagger}|3\rangle=|011\rangle.$$

Here 011 is the numeric three-bit representation of three, under the book’s leftmost-q_0 convention.

Checkpoint 1

PAUSE & THINK

For φ=3/8 and three counting bits, which numeric output is exact?

- ☐ 110
- ☐ 011
- ☐ 101

Answer: 011. The phase pattern is F8|3⟩. The inverse transform therefore gives label three with certainty.

Finite resolution is not hardware noise

Now choose phi equal to 0.3. It is not a multiple of one eighth. A perfect three-qubit QPE circuit therefore has several possible outputs. That spread exists in the ideal statevector; it does not imply decoherence or a broken gate. Increasing t refines the grid, while confidence procedures determine how repeated outputs are combined.

The amplitude for label j is an average of phase factors whose step depends on phi minus j over the grid size. At a matching grid point they all align. Away from a match they partly cancel. This finite geometric sum explains both the sharp exact-grid case and the broadened off-grid distribution.

Phase is periodic. Values near zero and near one represent nearby directions on the unit circle. An estimator should respect this wraparound when judging error. Treating 0.99 and 0.01 as far apart in an ordinary linear metric can misinterpret a phase estimate near the boundary.

The probability at each grid label comes from a finite sum of phase contributions.

$$c_j = \frac{1}{L}\sum_{k=0}^{L-1} e^{2\pi i k(\phi - j/L)}, \qquad P(j) = |c_j|^2, \qquad \widehat{\phi} = \frac{j}{L}.$$

For a grid-aligned phase $\phi = j_0/L$, the sum gives $P(j_0) = 1$. Off-grid phases give several possible labels even with exact gates.

Summing that geometric series gives a convenient exact plot formula.

$$P(j) = \frac{\sin^2(\pi L \Delta_j)}{L^2 \sin^2(\pi \Delta_j)}, \qquad \Delta_j = \phi - \frac{j}{L}.$$

At a zero denominator, use the continuous limit: one when Δ_j is an integer. For $t = 3$ and $\phi = 0.3$, the largest probabilities occur at $j = 2$ (about 0.5775) and $j = 3$ (about 0.2593).

What if the input is not an eigenstate?

Expand the work state in U's eigenbasis. The controlled operations attach the appropriate phase pattern to each component. Measurement of the counting register samples eigenphases with weights determined by the input overlaps, with additional finite-resolution effects. Conditional on a sufficiently informative result, the work register can be projected toward the corresponding eigenspace.

This behaviour can be useful, but it changes the task description. A tiny overlap with the desired eigenstate means a small chance of seeing its phase. Degeneracy means different eigenstates can share a phase. The algorithm does not automatically find a particular eigenvalue merely because U has one somewhere in its spectrum.

Complexity and implementation costs

With t counting qubits, the largest controlled power is exponential in t. If implemented by repeated U calls, the total scale is about two to the power t, matching an inverse-precision dependence. Efficiently computing a special power may reduce the cost in a structured application, but it must be justified. A logarithmic number of output qubits is not logarithmic evolution time.

The inverse QFT has polynomial gate cost in t, while controlled-U synthesis, work-state preparation and error correction may dominate. Approximation errors in U and its powers can shift the inferred phase. A resource budget must make those errors small enough for the desired output accuracy. QPE is a powerful subroutine precisely when these access and preparation requirements can be met usefully.

Checkpoint 2

PAUSE & THINK

If controlled powers are built from repeated U calls, how many base U uses do the t controls require in total?

- ☐ t
- ☐ 2t
- ☐ 2^t−1

Answer: 2^t−1. The inverse-precision work is hidden in controlled powers. A structured shortcut must be justified for the specific U.

Interpret an energy phase with care

In a Hamiltonian application, the unitary may be time evolution generated by an energy operator. An energy eigenvalue then determines a phase through the chosen evolution time and sign convention. QPE returns that phase modulo a full turn, not an unrestricted real energy automatically. Different energies can wrap to the same phase if the time is chosen without a suitable energy-range bound. The application must specify the scaling, possible shift and reconstruction interval. Increasing binary precision refines a phase estimate but does not remove an ambiguity caused by an incorrectly chosen range. This is another reason to state the full input and output task description before interpreting a number.

Predict and test

The lab uses three counting qubits and assumes an exact eigenstate. Predict label three for phase three eighths, then compare with the off-grid setting 0.3. The spread at the latter setting is finite resolution in an ideal model.

L08 · INTERACTIVE LAB

Runs on your device

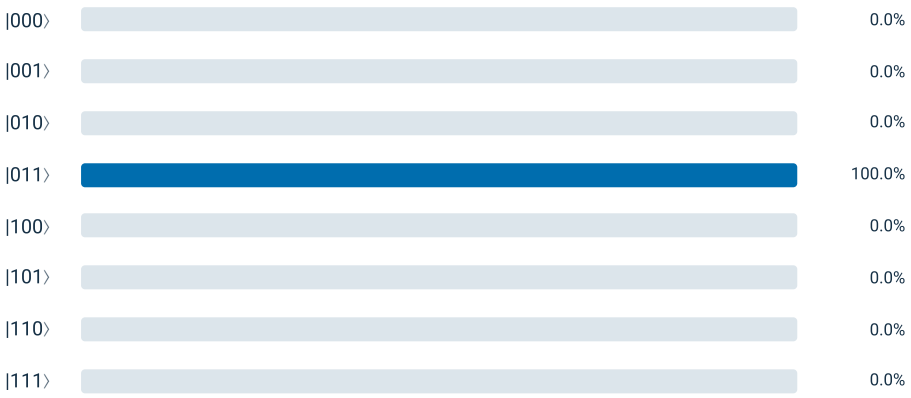
Quantum phase estimation

Three counting qubits estimate φ for an exact eigenstate with eigenvalue exp(2πiφ). Eigenstate preparation and controlled-power synthesis are assumed.

MODEL · Ideal pure-state vector, complex double precision ORDER · q0 is the leftmost bit

Initial setup · reference state

Exact probability



Statevector before readout · ideal simulation

Basis	Amplitude a + bi	Magnitude	Phase (rad)	Probability
|000⟩	0 + 0i	0.000000	undefined	0.0%
|001⟩	0 + 0i	0.000000	undefined	0.0%
|010⟩	0 + 0i	0.000000	undefined	0.0%
|011⟩	1 + 0i	1.000000	0.000000	100.0%
|100⟩	0 + 0i	0.000000	undefined	0.0%
|101⟩	0 + 0i	0.000000	undefined	0.0%
|110⟩	0 + 0i	0.000000	undefined	0.0%
|111⟩	0 + 0i	0.000000	undefined	0.0%

Signs and complex phases are part of the state. Phase is undefined for a zero amplitude. A shared global phase changes no probabilities; relative phases can affect a later operation. Values are rounded to six decimal places.

Prediction to try: Which three-bit estimate should be most likely? Is φ exactly on the 1/8 grid, or should several estimates occur?

Challenge: Compare φ = 3/8 with φ = 0.3. Explain finite resolution without calling it hardware noise.

Open the live experiment studio

Chapter summary

- An eigenstate turns controlled powers into a counting-register phase pattern.
- The inverse QFT converts an exact-grid pattern into its numeric phase label.

- Off-grid spread, preparation overlap and gate errors are separate effects.

A common misconception

Several QPE outcomes necessarily mean the hardware is noisy. A phase between grid points already produces a distribution in the ideal model. More counting bits refine the grid but also require higher controlled powers.

Code practice

Download the tested Qiskit notebook and follow the pinned environment instructions. Inspect the circuit, predict a result and run its independent assertions. The modular-action example is a small-instance demonstration; the notebook states its precision and post-processing limits.

Exercises

1. With three counting qubits and an exact phase of five eighths, what numeric label and bitstring should appear?
2. A three-qubit run at phase 0.3 gives several nonzero output probabilities in an ideal simulator. What should you change to improve resolution, and what additional cost does that introduce?

<details>
<summary>Worked answers</summary>

1. The label is five and the bitstring is 101 in numeric leftmost-most-significant order. The inverse QFT recognises the exact phase-grid pattern.
2. Add counting precision or use an appropriate estimation variant and confidence strategy. In the canonical circuit, greater precision requires higher controlled powers, so coherent work and reliability demands increase. Merely increasing final measurement shots does not change the three-bit grid itself.

</details>

Check your understanding

PAUSE & THINK

An ideal three-bit QPE run at phase 0.3 has several possible outputs. Why?

- [ ] The phase is off the 1/8 grid
- [ ] Every ideal gate is noisy
- [ ] The simulator forgot normalisation

Answer: The phase is off the 1/8 grid. Finite counting resolution produces a spread even with perfect operations. More counting precision refines the grid but requires higher controlled powers.

Sources

IBM's phase-estimation problem statement states the eigenstate input task description. Its procedure lesson derives the controlled-power and inverse-QFT construction.