Qubitization

Relate normalised Hamiltonian eigenvalues to qubitization angles and target evolution.

Research 22 min lesson Dr. Charith Jayasekara

By the end, you'll be able to

- Specify signal access and normalisation.
- Calculate arccosine angles for a two-term Hamiltonian.
- Explain why a phase-processing sequence is still needed.

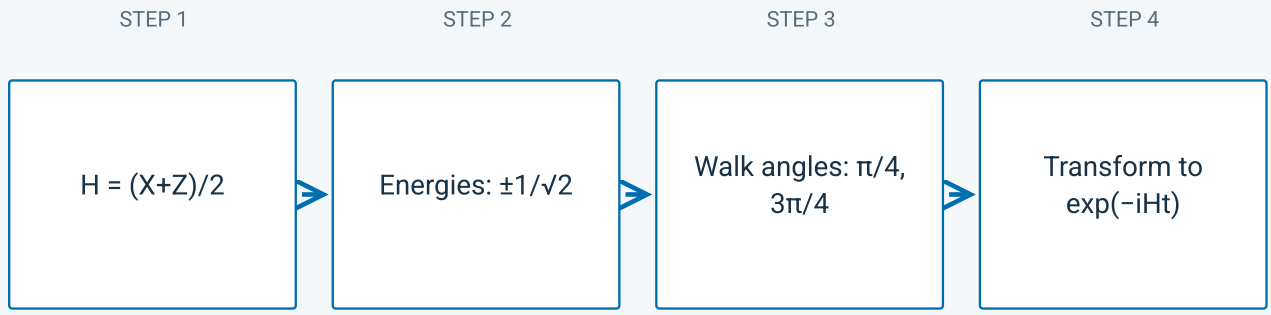
Turn an eigenvalue into a rotation angle

Qubitization is a technique that organises a Hamiltonian's action into two-dimensional invariant subspaces. Within each subspace, a constructed unitary behaves like a rotation whose angle is related to a normalised energy eigenvalue. Once that relationship is available, carefully chosen phase sequences can implement functions of the energy, including time evolution.

Prerequisites: unitaries and matrices, phase and interference, LCU and Hamiltonian simulation. The original qubitization research establishes an efficient oracle-based framework and its simulation consequences. Low and Chuang.

WORKED DIAGRAM · A25

Energy-to-angle mapping



The walk angle arccos(λ) is not the physical-time phase −λt. Qubitization supplies a structured signal for a further transformation.

The signal-access setup

Assume a unitary and a designated ancilla state provide a block equal to H divided by a positive scale alpha. This normalised Hamiltonian has eigenvalues between negative one and one. The preparation and selection routines, their

inverses and any controlled versions required by the construction must be available at stated cost. These are substantive assumptions, not free access to an arbitrary large matrix.

For a convenient Hermitian-unitary signal construction, combine the signal with a reflection about the designated ancilla subspace to obtain a walk operator. In each energy sector, its eigenphases are related to plus or minus the arccosine of the normalised eigenvalue. More general block encodings require the appropriate qubitization construction; a bare arbitrary block is not automatically the same walk.

Worked example: two Pauli terms

Let H equal one half X plus one half Z, with alpha equal to one. Its squared value is one half times the identity because X and Z anticommute. The two energy eigenvalues are therefore positive and negative one over square root of two. These are within the required unit interval.

An LCU signal uses an ancilla in plus and SELECT choosing X or Z. Both selected terms are Hermitian involutions, giving the convenient signal structure. The associated rotation angles are arccosine of positive one over square root of two, equal to pi over four, and arccosine of its negative, equal to three pi over four.

If the desired evolution time is t, the target physical phases on the two energy eigenvectors are the exponential of negative i times t divided by square root of two and its conjugate. Merely applying one walk step gives phases related to arccosine instead, so qubitization alone is not the final time evolution. A signal-processing sequence must transform the spectral response into the desired function.

The signal walk turns a normalised energy into an angle, while physical time evolution uses the energy itself.

$$H|E\rangle = E|E\rangle, \qquad \cos\theta_E = \frac{E}{\alpha}, \qquad \lambda_{\mathrm{walk}} = e^{\pm i\theta_E}, \qquad e^{-iHt}|E\rangle = e^{-iEt}|E\rangle.$$

α is the signal normalisation. This is the reflection-times-signal walk convention in which the relevant two-dimensional eigenphases are $\pm\arccos(E/\alpha)$; signal processing is needed to turn that response into the desired time-evolution phase.

The chapter's two eigenvalues give two simple angle pairs.

$$H = \frac{X+Z}{2}, \quad \alpha = 1, \qquad E_{\pm} = \pm\frac{1}{\sqrt{2}}, \qquad \theta_{+} = \frac{\pi}{4}, \quad \theta_{-} = \frac{3\pi}{4}.$$

The subscript labels the physical energy sign. Each energy sector has the associated positive and negative walk eigenphase, so one walk step is generally different from evolution for a chosen time t.

Why this organisation helps

A high-dimensional Hamiltonian may have many eigenvectors, but the same sequence acts on each associated two-dimensional signal subspace according to its eigenvalue. The algorithm therefore implements a scalar function simultaneously across the spectrum without measuring which eigenvalue is present. Measuring the phase midway would change the coherent task.

The difficult design problem becomes polynomial or phase-sequence approximation under boundedness and symmetry conditions. That does not remove all complexity, but it gives a systematic framework with strong query guarantees. The selected sequence length depends on the scale-time product and desired precision, as well as the precise function and access model.

A step-by-step conceptual procedure

First choose a representation and normalisation scale for H. Build a tested signal or block encoding. Construct the walk and verify its spectral relation on small matrices. Design a polynomial approximation to the desired response over the spectrum. Find a compatible phase sequence and implement it using repeated signal accesses and ancilla rotations. Finally measure the application observable with a separate sampling budget.

For the toy Hamiltonian, exact diagonalisation supplies a complete reference. Compare the two expected spectral phases and the resulting state evolution on a superposition of eigenvectors. Checking only an eigenstate's Z probabilities could miss a wrong phase because an eigenstate evolves only by a common factor. A superposition is a more informative functional test.

Normalisation is a resource

If the block encodes H divided by alpha, simulating physical time t involves a dimensionless scale alpha times t. An unnecessarily large alpha can make a query-efficient-looking representation less useful. Different decompositions of the same Hamiltonian may lead to different normalisation and preparation costs.

Ancillas, controlled SELECT, precision of rotations and fault-tolerant synthesis also contribute to physical cost. A favourable oracle-query theorem does not by itself specify a near-term hardware implementation. Qubitization is particularly important in fault-tolerant algorithm design, where long coherent circuits and carefully costed non-Clifford operations are considered explicitly.

Test phases with a superposed energy input

An energy eigenstate acquires only a common phase under physical evolution, so its ordinary probabilities may not reveal a wrong phase function. Prepare an equal superposition of the toy Hamiltonian's positive- and negative-energy eigenvectors. Their relative phase after time t should change by square root of two times t.

A walk step instead uses the arccosine-derived phases. Confusing that walk with the target evolution can therefore pass an eigenstate-probability test while failing on the superposition. Interference after an appropriate basis change exposes the error. This is another example of choosing a test input sensitive to the distinction being claimed.

For a small matrix, compare the complete unitary or several independent input states against exact exponentiation. Preserve a separate check of ancilla leakage as well as the conditional system response. A normalised output can look correct while an unexpectedly small success amplitude makes the full implementation costly.

Limits and connections

Qubitization does not prepare a desired ground state automatically or print a full spectrum. It implements a structured access transformation that other algorithms use. State preparation, spectral filtering, phase estimation and observable extraction remain distinct tasks. Its efficiency depends on a useful signal representation being available.

The subsequent QSVT framework generalises polynomial transformations to singular values and supplies a common language for several matrix algorithms. Gilyén and colleagues. It is useful to learn the two-dimensional rotation picture first, then study the more general algebra after the input and output task descriptions are clear.

Exercises and worked answers

Exercise 1. What qubitization angle corresponds to normalised energy zero? What about positive one?

Check the worked answer

Arccosine zero is pi over two. Arccosine one is zero. The angle-energy mapping is nonlinear, which is why a walk step is not generally the same as the desired physical-time evolution.

Exercise 2. A decomposition doubles alpha while leaving the physical Hamiltonian and time fixed. What resource concern follows?

Check the worked answer

The scale-time product doubles, potentially increasing the required signal-query degree. Compare any simplification in PREPARE or SELECT against that increase. Normalisation and subroutine implementation costs must be assessed together.

Check your understanding

PAUSE & THINK

Which walk angle corresponds to normalised energy zero?

- [ ] 0
- [ ] π/2
- [ ] π

Answer: π/2. The angle is arccos(0) = π/2. The nonlinear mapping is why one walk step is not generally the target time evolution.

Sources

- Low and Chuang
- Gilyén and colleagues