Superdense coding

Encode two classical bits using a shared Bell pair and one transmitted qubit.

Core 18 min lesson Dr. Charith Jayasekara

By the end, you'll be able to

- Encode and decode two bits using a Bell pair.
- Account for the transmitted qubit and previously shared entanglement.
- Distinguish joint information from local measurement statistics.

Before you begin: Bell states, H, X, Z, CNOT and measurement.

Count the resources first

Alice wants to send Bob one of four messages: 00, 01, 10 or 11. An ordinary binary channel needs two bits to distinguish them. Superdense coding uses one transmitted qubit, provided Alice and Bob already share a Bell pair. The pre-shared pair is part of the cost. This is a precise example of entanglement changing what a communication task can accomplish.

Start with Alice's qubit A and Bob's qubit B in

$$|\Phi^+\rangle_{AB} = \frac{|00\rangle + |11\rangle}{\sqrt{2}}.$$

Throughout this guide, the written basis order is **Alice first, Bob second**. This matters because software often prints bit strings in a different order. Fixing the convention before calculating prevents a correct circuit from appearing to return reversed bits.

Encode two choices in one shared state

Let the message be pq, where p is the first bit. Alice applies X^q and then Z^p to her qubit. A zero exponent means "skip this gate"; a one exponent means "apply it". Her operation is therefore Z^pX^q.

The two bits select the correlation and phase of the pair:

$$|\beta_{pq}\rangle = (Z^pX^q \otimes I)|\Phi^+\rangle.$$

Message pq	Alice's operation	Shared state
00	I	$(
01	X	$(
10	Z	$(
11	ZX	$(

These four Bell states are mutually orthogonal. Bob can distinguish them when he has access to **both** qubits. Alice sends A to Bob, who now decodes the joint state.

Checkpoint. What changes when p changes while $q = 0$? The relative sign between $|00\rangle$ and $|11\rangle$. Their Z-basis probabilities remain equal, so simply measuring the two wires immediately would lose the phase bit.

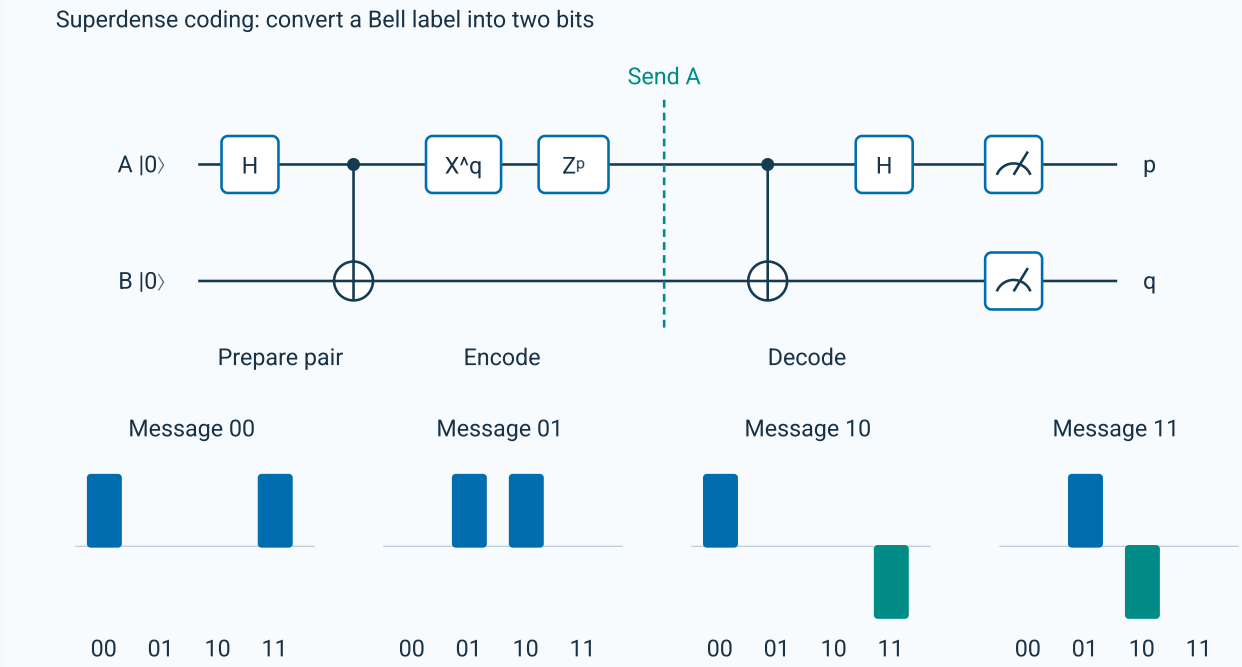
Decode with an actual circuit

Bob applies CNOT with A as control and B as target, then applies H to A. In the chosen basis order, the result is

$$\mathrm{CNOT}_{A\to B}\,|\beta_{pq}\rangle = \frac{|0q\rangle + (-1)^p|1q\rangle}{\sqrt{2}},$$

$$(H_A \otimes I_B)\,\mathrm{CNOT}_{A\to B}\,|\beta_{pq}\rangle = |pq\rangle.$$

The first output gives the phase bit p; the second gives the correlation bit q. The Hadamard acts like the second half of an interferometer: it makes the relative sign visible in an ordinary measurement outcome.



Superdense coding circuit and signed amplitudes of the four encoded Bell states.

The four panels show signed amplitudes in basis order 00,01,10,11. Blue bars are positive and teal bars negative. Comparing messages 00 and 10 reveals the phase change that the decoding circuit converts into a measured bit.

Worked message: 10

For message 10, Alice applies Z and obtains

$$\frac{|00\rangle - |11\rangle}{\sqrt{2}}.$$

CNOT maps $|00\rangle$ to itself and $|11\rangle$ to $|10\rangle$. The state becomes $(|00\rangle - |10\rangle)/\sqrt{2} = |-\rangle_A|0\rangle_B$. Because $H|-\rangle = |1\rangle$, decoding yields $|10\rangle$ with certainty in the ideal circuit. The phase difference has become the first output bit.

Explore freely, then predict

Select any message and inspect the state after each gate. Keep signed amplitudes visible in the simulator. First compare messages 00 and 10: the middle probabilities are identical, while the amplitude signs differ. Then reveal the decoded outputs. A useful quick prediction is “upper bit changes”, “lower bit changes” or “both change” when toggling one message bit. Exploration can continue without entering a written prediction each time.

Now remove Bob's H and repeat the four messages. He can still learn q from parity, but the computational-basis statistics do not distinguish the two possible values of p. This locates the essential interference operation rather than treating the protocol as a memorised gate list.

Bob's original qubit alone has state $I/2$ for every message. Alice's local encoding leaves that marginal state unchanged. Bob gains the two-bit information when he receives the transmitted qubit and performs a joint decoding operation. Preparing and distributing the entangled pair must therefore be included when comparing communication resources or practical performance.

Exercises

1. Decode message 01 by hand.
2. What would a gate-order change do for message 11?

<details>
<summary>Worked answers</summary>

1. CNOT transforms $(|01\rangle + |10\rangle)/\sqrt{2}$ into $|+\rangle_A|1\rangle_B$; H_A yields $|01\rangle$.
2. $XZ = -ZX$, so reversing those two gates changes only the overall sign for that message. Its measurement outcome stays 11. Keep the convention fixed to make the algebra easier to follow.

</details>

Check your understanding

PAUSE & THINK

Why can Bob recover two bits?

- [ ] He receives one qubit and uses a previously shared entangled pair
- [ ] One isolated qubit contains two directly readable classical bits
- [ ] Entanglement sends Alice's message before transmission

Answer: He receives one qubit and uses a previously shared entangled pair. Both resources belong in the accounting.

The protocol uses four distinguishable joint states. Local gates choose a state, transmission brings the two halves together, and interference reveals the message.

Sources

Bennett and Wiesner, original dense-coding paper, 1992; IBM Quantum Learning, superdense coding. The message convention, worked derivation and exploration above are original.